Weight-Two Modular Calabi-Yau Manifolds From Permutation Symmetry

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Based on [2302.03047], in collaboration with
Philip Candelas, Xenia de la Ossa, and Pyry Kuusela

Overview

- We will discuss a restriction on a Calabi-Yau manifold $\mathcal{X}$'s complex structure, so that $\mathcal{X}$ is *weight-two modular*.

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- By developing methods for quickly calculating the zeta function when $\mathcal{X}$ is multiparameter, we are able to produce extensive tables of data to support our claim. We expect these methods to see use beyond this project.
- For illustration, our examples will be mirrors of favourable CICY manifolds. However, the story is much more general than this.

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If we consider $\mathcal{X}_\varphi$ as a variety over a finite field $\mathbb{F}_{p^n}$ for p prime, we will find that this variety consists of $\#_{p^n}(\varphi)$ points.

Now fix a prime p and collect these point-counts into the exponentiated generating function

$$\zeta_p(\varphi; T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#_{p^n}(\varphi)}{n} \cdot T^n\right) .$$

Weil gave the remarkable conjecture that the zeta function so defined is actually a rational function of T, with the form

$$\zeta_p(\varphi;T) = \frac{R_p(\varphi;T)}{(1-T)(1-pT)^{h^{1,1}}(1-p^2T)^{h^{1,1}}(1-p^3T)} ,$$

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Our discussion will turn to a restriction in φ-space, so that R_p possesses a particular property: *persistent factorisation*.

It may happen that for some φ_*, the polynomials $R_p(\varphi_*;T)$ have for every[1] prime p a degree-2 factor:

$$R_p(\varphi_*;T) = (1 - \alpha_p\, pT + p^3T^2)\tilde{R}_p\ , \qquad \deg_T\left(\tilde{R}\right) = 2h^{2,1}\ .$$

[1] Up to the subtlety of bad primes

(weight-two) Calabi-Yau modularity

It may happen that for some φ_*, the polynomials $R_p(\varphi_*;T)$ have for every[2] prime p a degree-2 factor:

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Further, α_p is the coefficient of q^p in the q-expansion of some weight-two modular newform .

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It may happen that for some φ_*, the polynomials $R_p(\varphi_*;T)$ have for every[3] prime p a degree-2 factor:

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In this case the manifold $\mathcal{X}_\varphi$ is said to be *weight-two modular*.

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It may happen that for some φ_*, the polynomials $R_p(\varphi_*; T)$ have for every[4] prime p a degree-2 factor:

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Further, α_p is the coefficient of q^p in the q-expansion of some weight-two modular newform .

In this case the manifold $\mathcal{X}_\varphi$ is said to be *weight-two modular*.

There is a question:

For which values φ_* do we have weight-two modularity?

[4] Up to the subtlety of bad primes

A (nonexhaustive) survey of CY modularity

[Yui 2011, references therein] A review up to 2011.

[Hulek, Verrill, 2005] Proved weight-four modularity for a number of manifolds associated to the A4 lattice.

[Gouveau, Yui, 2009] Proved weight-four modularity of *rigid* threefolds defined over $\mathbb{Q}$.

[Candelas, Elmi, de la Ossa, van Straten, 2019] Computed tables of zeta functions for the HV family, identifying a *rank-two attractor*: a nonsingular threefold with weight-four modularity.

[Bönisch, Klemm, Scheidegger, Zagier, 2022] Studied the one-parameter hypergeometric families, exhibiting modularity at conifolds and new rank-two attractors. Provided a proven example, by constructing a modular parametrisation (see also Bönisch's talk in this series).

[Bönisch, Elmi, Kashani-Poor, Klemm, 2022] Gave a number of new examples of rank-two attractors.

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We study both of these problems, and so provide a large number of new examples supporting the conjecture.

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Flux compactifications

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This potential term is built out of a superpotential

$$W \;=\; \int_{\mathcal{X}} (F_3 - \tau H_3) \wedge \Omega \;=\; (F - \tau H) \cdot \Sigma \cdot \Pi \;,$$

with Σ being the standard symplectic form $\Sigma = \begin{pmatrix} 0 & \mathbb{I}_{h^{2,1}} \\ -\mathbb{I}_{h^{2,1}} & 0 \end{pmatrix}$.

Supersymmetric flux vacua (SFV)

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The problem is to find pairs of flux vectors F, H and values for the moduli φ^i and axiodilaton τ that solve the above equations.

In many cases, an exchange of two moduli $\varphi^j \leftrightarrow \varphi^k$ will swap pairs of components of the period vector:

$$\Pi^j \leftrightarrow \Pi^k \ , \qquad \Pi^{h^{2,1}+1+j} \leftrightarrow \Pi^{h^{2,1}+1+k} \ .$$

So as to refer back to this, call this property $\mathcal{S}$.

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If our compactification manifold $\mathcal{X}$ has the property $\mathcal{S}$, then one can solve the SFV equations by choosing fluxes

$$F \ = \ \mathbf{e}_{(i)} - \mathbf{e}_{(k)} \ , \qquad H \ = \ \mathbf{e}_{(h^{2,1}+1+k)} - \mathbf{e}_{(h^{2,1}+1+j)} \ ,$$

and constraining the moduli to the invariant locus $\boxed{\varphi^j = \varphi^k.}$

The vectors $\mathbf{e}_{(i)}$ are the standard orthonormal basis of $\mathbb{R}^{2h^{1,2}+2}$.

With this choice of fluxes and moduli, all but two of the SFV equations are immediately solved.

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The remaining equations are

$$(F-\tau H)\cdot\Sigma\cdot\partial_{\varphi^j}\Pi\Big|_{\varphi^j=\varphi^k}=0\,,\qquad (F-\tau H)\cdot\Sigma\cdot\partial_{\varphi^k}\Pi\Big|_{\varphi^j=\varphi^k}=0\,.$$

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By the $\mathcal{S}$ property, these equations are actually the same condition. To solve them, set

$$\tau = \frac{F \cdot \Sigma \cdot \partial_{\varphi^j} \Pi}{H \cdot \Sigma \cdot \partial_{\varphi^j} \Pi} \Bigg|_{\varphi^j = \varphi^k}\,.$$

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τ is *ab initio* a function of the $h^{2,1}-2$ unconstrained moduli and the shared value of $\varphi^j = \varphi^k = \theta$. In several cases the θ dependence drops out.

CICY Manifolds (fixing notation)

(Following [**Candelas, Dale, Lütken, Schimmrigk, 1988**])

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Consider the configuration $\begin{matrix}\mathbb{P}^1\\ \mathbb{P}^3\end{matrix}\begin{bmatrix}2\\ 4\end{bmatrix}$, which specifies the vanishing locus in $\mathbb{P}^1 \times \mathbb{P}^3$ of a polynomial with degrees 2 and 4 in each factor of the ambient space.

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Consider also the configuration $\begin{matrix}\mathbb{P}^3 \\ \mathbb{P}^3\end{matrix}\begin{bmatrix}1 & 1 & 2 \\ 1 & 2 & 1\end{bmatrix}$, which specifies the vanishing locus of three polynomials in $\mathbb{P}^3 \times \mathbb{P}^3$ with multidegrees $(1,1)$, $(1,2)$, and $(2,1)$.

More generally we can look at an intersection of c hypersurfaces

$$\begin{matrix} \mathbb{P}^{n_1} \\ \vdots \\ \mathbb{P}^{n_k} \end{matrix} \begin{bmatrix} d_{1,1} & ... & d_{1,c} \\ \vdots & ... & \vdots \\ d_{k,1} & ... & d_{k,c} \end{bmatrix},$$

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Most such threefolds are **not** Calabi-Yau. You get a Calabi-Yau if you have $\sum_{a=1}^{c} d_{i,a} = n_i + 1$.

We shall for now only consider CICY manifolds $\mathcal{Y}$ whose second cohomology $H^2(\mathcal{Y}, \mathbb{Z})$ is generated by the pullbacks to $\mathcal{Y}$ of the Kähler classes K_j of the ambient factors $\mathbb{P}^{n_j}$.

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A swapping of the ambient factors $\mathbb{P}^{n_j}$ and $\mathbb{P}^{n_k}$ thereby effects a swap of a pair of $\mathcal{X}$'s complex structure moduli.

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The holomorphic period is

$$\varpi_0 = \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{\prod_{a=1}^{c}\left(\sum_{b=1}^{k} d_{b,a} m_b\right)!}{\prod_{b=1}^{k}(m_b!)^{n_b+1}} \prod_{b=1}^{k}\left(\varphi^b\right)^{m_b} \equiv \sum_{\mathbf{m}\geq 0} c\left(\mathbf{m}\right)\varphi^{\mathbf{m}}\,.$$

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Obviously (?) if the configuration matrix is symmetric under the exchange of the i^{th} and j^{th} rows, then ϖ_0 is a symmetric function of φ^i and φ^j .

The triple intersection numbers Y_{ijk} can be computed as the coefficient of the volume form in the expansion of

$$e_i e_j e_k \prod_{a=1}^{c} \left(1 + \sum_{b=1}^{k} d_{b,a} e_b\right) \ .$$

The period vector

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A symmetry between the j, k rows of the configuration matrix gives rise to $Y_{ijk} = Y_{ikj}$.

The remaining $h^{2,1}$ logarithmic Frobenius periods, $h^{2,1}$ log-squared periods, and the final log-cubed period are found by taking

$$\varpi_{1,i} \;=\; \partial_{\epsilon_i}\varpi^{\epsilon}\Big|_{\epsilon=0}\,,$$

$$\varpi_{2,i} \;=\; \frac{1}{2}Y_{ijk}\partial_{\epsilon_j}\partial_{\epsilon_k}\varpi^{\epsilon}\Big|_{\epsilon=0}\,,$$

$$\varpi_{3} \;=\; \frac{1}{6}Y_{ijk}\partial_{\epsilon_i}\partial_{\epsilon_j}\varpi^{\epsilon}\Big|_{\epsilon=0}\,,$$

with $\varpi^{\epsilon} \;\equiv\; \sum_{\mathbf{m}\geq 0}\frac{c(\mathbf{m}+\epsilon)}{c(\epsilon)}\varphi^{\mathbf{m}+\epsilon}$

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where $\nu = \text{diag}(1, 2\pi\text{i}\,\mathbf{1}, (2\pi\text{i})^2\,\mathbf{1}, (2\pi\text{i})^3)$ and

$$\rho \;=\; \begin{pmatrix} -\frac{1}{3}Y_{000} & -\frac{1}{2}\mathbf{Y}_{00}^T & \mathbf{0}^T & 1 \\ -\frac{1}{2}\mathbf{Y}_{00} & -\mathbb{Y}_0 & -\mathbb{I} & \mathbf{0} \\ 1 & \mathbf{0}^T & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{I} & \mathbb{0} & 0 \end{pmatrix}.$$

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This matrix contains the topological data

$$Y_{00i} \;=\; -\frac{1}{12}\int_{\mathcal{X}} c_2\wedge e_i \;, \quad Y_{000} \;=\; \frac{3\chi(\mathcal{X})\zeta(3)}{(2\pi\mathrm{i})^3} \;, \quad Y_{0ij} \in \left\{0, \frac{1}{2}\right\} \;.$$

The punchline

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We make a comment on the form of the axiodilaton:

$$\tau(\varphi) \; = \; \frac{\mathrm{i}}{2\pi} \, \frac{\partial_{\varphi^i} \left(\varpi_i - \varpi_j \right)}{\partial_{\varphi^i} \left(\varpi^i - \varpi^j \right)} \bigg|_{\varphi^i = \varphi^j} + Y_{0ij} - Y_{0ii}$$

The mirrors of the following manifolds possess $\mathcal{S}$:

$$\begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&1&1&1\\1&1&1&1&1\end{bmatrix}, \quad \begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}3\\3\end{bmatrix}, \quad \begin{matrix}\mathbb{P}^3\\ \mathbb{P}^3\end{matrix}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}, \quad \begin{matrix}\mathbb{P}^3\\ \mathbb{P}^3\end{matrix}\begin{bmatrix}1&1&2\\1&2&1\end{bmatrix},$$

$$\begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}, \quad \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}2\\2\\3\end{bmatrix}, \quad \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}2\\2\\2\\2\end{bmatrix}, \quad \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}1&1\\1&1\\1&1\\1&1\\1&1\end{bmatrix}.$$

Counter example: the mirror to $\begin{matrix}\mathbb{P}^1\\ \mathbb{P}^3\end{matrix}\begin{bmatrix}2\\4\end{bmatrix}$ does not possess $\mathcal{S}$.

So far we have computed tables of modular forms for the first two and final families in the above list.

Example 1

First, consider the five-parameter mirror to $\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{matrix}\left[\begin{matrix}1 & 1\\1 & 1\\1 & 1\\1 & 1\\1 & 1\end{matrix}\right]$.

The periods (in the LCS region) are

$$
\begin{aligned}
\varpi^0(\varphi) &= \int_0^\infty \mathrm{d}z\; z\,\mathrm{K}_0(z)\prod_{i=1}^{5}\mathrm{I}_0\left(\sqrt{\varphi^i}\,z\right)\;,\\
\varpi^j(\varphi) &= -2\int_0^\infty \mathrm{d}z\; z\,\mathrm{K}_0(z)\mathrm{K}_0\left(\sqrt{\varphi^j}\,z\right)\prod_{i\neq j}\mathrm{I}_0\left(\sqrt{\varphi^i}\,z\right)\;,\\
\varpi_j(\varphi) &= 8\sum_{\substack{m<n\\ m,n\neq j}}\int_0^\infty \mathrm{d}z\; z\,\mathrm{K}_0(z)\mathrm{K}_0\left(\sqrt{\varphi^m}\,z\right)\mathrm{K}_0\left(\sqrt{\varphi^n}\,z\right)\prod_{i\neq m,n}\mathrm{I}_0\left(\sqrt{\varphi^i}\,z\right) - 4\pi^2\varpi_0(\varphi)\;.
\end{aligned}
\tag{1}
$$

Example 1

First, consider the five-parameter mirror to $\begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\end{bmatrix}$.

To get an SFV, set $\varphi^4 = \varphi^5$. The axiodilaton is

$$\tau\left(\varphi^1,\, \varphi^2,\, \varphi^3\right) \;=\; \frac{2\mathrm{i}}{\pi} \cdot \frac{\int_0^\infty \mathrm{d}z\; z\, \mathrm{K}_0(z) \left[\mathrm{K}_0\left(\sqrt{\varphi^1}z\right)\, \mathrm{I}_0\left(\sqrt{\varphi^2}z\right)\, \mathrm{I}_0\left(\sqrt{\varphi^3}z\right) \;+ \text{cyclic}\right]}{\int_0^\infty \mathrm{d}z\; z\, \mathrm{K}_0(z)\, \mathrm{I}_0\left(\sqrt{\varphi^1}z\right)\, \mathrm{I}_0\left(\sqrt{\varphi^2}z\right)\, \mathrm{I}_0\left(\sqrt{\varphi^3}z\right)}\;.$$

This function τ satisfies

$$j\left(\tau\left(\varphi^1,\, \varphi^2,\, \varphi^3\right)\right) \;=\; \frac{\left(\Delta_F + 16\varphi^1\varphi^2\varphi^3\right)^3}{\Delta_F\left(\varphi^1\varphi^2\varphi^3\right)^2}\ ,$$

where

$$\Delta_F = \left(\left(1-\varphi^1-\varphi^2-\varphi^3\right)^2 - 4\left(\varphi^1\varphi^2+\varphi^2\varphi^3+\varphi^3\varphi^1\right)\right)^2 - 64\,\varphi^1\varphi^2\varphi^3\ .$$

This same j-invariant appears in

[Verrill, 2004] and **[Bloch, Kerr, Vanhove, 2016]**.

Tables for example 1 (with LMFDB labels)

φ	Modular form label	φ	Modular form label	φ	Modular form label	φ	Modular form label
-64	75010.2.a.k	$-\frac{79}{2}$	337962.2.a.d	-21	87780.2.a.t	$-\frac{21}{2}$	184506.2.a.f
-63	2982.2.a.j	-39	4290.2.a.p	$-\frac{61}{3}$	25254.2.a.q	$-\frac{41}{4}$	458790.2.a.g
-61	416020.2.a.b	-38	10374.2.a.l	$-\frac{81}{4}$	373830.2.a.h	-10	10010.2.a.o
-59	470820.2.a.o	-37	469604.2.a.a	-20	38010.2.a.ba	$-\frac{29}{3}$	5742.2.a.t
-56	402990.2.a.cj	-36	14430.2.a.bj	$-\frac{77}{4}$	322014.2.a.bh	$-\frac{19}{2}$	138054.2.a.j
-55	23870.2.a.b	-35	33180.2.a.r	-19	16340.2.a.c	$-\frac{28}{3}$	332010.2.a.cx
-54	160710.2.a.w	-34	365330.2.a.k	$-\frac{56}{3}$	96642.2.a.bx	-9	2460.2.a.c
-53	152004.2.a.g	$-\frac{101}{3}$	449046.2.a.d	-18	18582.2.a.l	$-\frac{17}{2}$	100130.2.a.a
-51	304980.2.a.r	-33	334356.2.a.e	$-\frac{53}{3}$	33390.2.a.f	$-\frac{25}{3}$	23940.2.a.s
-50	230010.2.a.br	$-\frac{131}{4}$	357630.2.a.be	$-\frac{69}{4}$	50370.2.a.h	-8	438.2.a.g
$-\frac{149}{3}$	356706.2.a.i	-32	1122.2.a.j	-17	15708.2.a.g	$-\frac{23}{3}$	376740.2.a.v
-49	30940.2.a.g	-31	2170.2.a.k	$-\frac{49}{3}$	121212.2.a.n	$-\frac{15}{2}$	69870.2.a.k
$-\frac{97}{2}$	224070.2.a.bc	$-\frac{121}{4}$	120230.2.a.b	-16	4930.2.a.e	$-\frac{22}{3}$	66330.2.a.bn
-48	18186.2.a.e	-30	252030.2.a.p	$-\frac{46}{3}$	402822.2.a.bd	-7	14.2.a.a
-47	14946.2.a.m	-29	227940.2.a.f	-15	510.2.a.g	$-\frac{27}{4}$	45942.2.a.r
-45	280140.2.a.bf	$-\frac{85}{3}$	16830.2.a.w	$-\frac{29}{2}$	472874.2.a.a	$-\frac{20}{3}$	126270.2.a.bk
$-\frac{133}{3}$	203490.2.a.t	-28	102718.2.a.p	-14	26670.2.a.bf	$-\frac{13}{2}$	46410.2.a.bf
-44	131010.2.a.i	-27	5124.2.a.b	$-\frac{27}{2}$	6090.2.a.j	$-\frac{19}{3}$	218196.2.a.i
-43	183524.2.a.a	-26	18330.2.a.w	$-\frac{40}{3}$	42570.2.a.bd	$-\frac{25}{4}$	66410.2.a.a
$-\frac{125}{3}$	4230.2.a.bb	$-\frac{77}{3}$	200970.2.a.eu	-13	21476.2.a.c	-6	2310.2.a.t
$-\frac{81}{2}$	364038.2.a.e	-25	29380.2.a.b	$-\frac{25}{2}$	6810.2.a.e	$-\frac{23}{4}$	29118.2.a.e
-40	7790.2.a.d	$-\frac{49}{2}$	316302.2.a.g	$-\frac{37}{3}$	23310.2.a.w	$-\frac{17}{3}$	39780.2.a.x

The HV manifold is birational to the intersection in $\mathbb{T}^5$:

$$X_0 + X_1 + X_2 \;=\; -\left(X_3 + X_4 + X_5\right)\;,$$

$$\frac{\varphi^0}{X_0} + \frac{\varphi^1}{X_1} + \frac{\varphi^2}{X_2} \;=\; -\left(\frac{\varphi^3}{X_3} + \frac{\varphi^4}{X_4} + \frac{\varphi^5}{X_5}\right)\;,$$

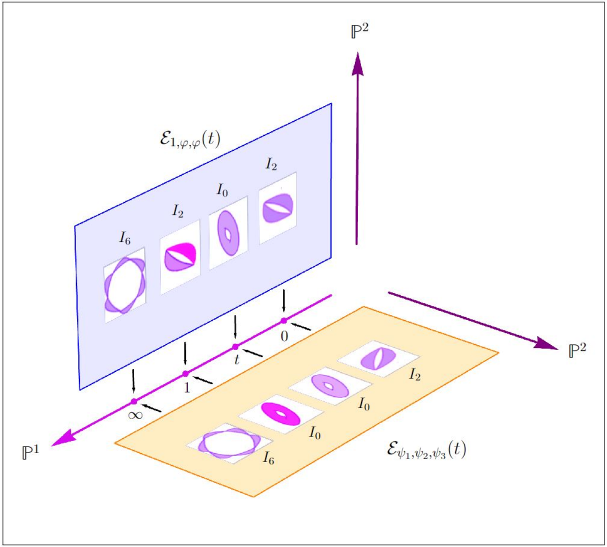
As a consequence of these relations, we can write

$$(X_0 + X_1 + X_2)\left(\frac{\varphi^0}{X_0} + \frac{\varphi^1}{X_1} + \frac{\varphi^2}{X_2}\right)t_0 \;=\; t_1\;,$$

$$(X_3 + X_4 + X_5)\left(\frac{\varphi^3}{X_3} + \frac{\varphi^4}{X_4} + \frac{\varphi^5}{X_5}\right)t_0 \;=\; t_1\;, \qquad (t_0 : t_1) \in \mathbb{P}^1\;,$$

And so a fibred product
$\mathcal{E}_{\varphi^0,\varphi^1,\varphi^2}(t) \times_{\mathbb{P}^1} \mathcal{E}_{\varphi^3,\varphi^4,\varphi^5}(t) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ birational to the Hulek-Verrill manifold is found.

The geometry of example 1



Example 2

Consider the two-parameter mirror to $\begin{matrix}\mathbb{P}^4 \\ \mathbb{P}^4\end{matrix}\begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1\end{bmatrix}$.

Set both moduli equal to φ. The axiodilaton is a ratio of integrals of products of Meijer G functions, and the j-invariant of this is

$$j(\tau(\varphi)) = \frac{(1 + 12\varphi + 14\varphi^2 - 12\varphi^3 + \varphi^4)^3}{\varphi^5(\varphi^2 - 11\varphi - 1)} .$$

Incidentally, this model also has a rank-two attractor at $\varphi^1 = \varphi^2 = -1$, to which we will return in ongoing work.

Tables for example 2

φ	Modular form label	φ	Modular form label	φ	Modular form label	φ	Modular form label
-100	**110990.2.a.j**	-67	**350075.2.a.a**	-38	**70718.2.a.a**	-10	**2090.2.a.l**
-99	**359337.2.a.a**	-66	**335346.2.a.f**	-37	**65675.2.a.b**	$-\frac{49}{5}$	**177485.2.a.a**
-98	**149534.2.a.d**	-65	**321035.2.a.b**	-36	**10146.2.a.p**	$-\frac{39}{4}$	**251238.2.a.f**
-96	**61626.2.a.h**	-64	**9598.2.a.b**	-35	**56315.2.a.a**	$-\frac{48}{5}$	**147570.2.a.i**
-92	**435850.2.a.j**	-63	**97881.2.a.b**	-34	**51986.2.a.e**	$-\frac{19}{2}$	**29450.2.a.r**
-90	**272670.2.a.p**	-62	**280550.2.a.h**	-33	**47883.2.a.a**	$-\frac{37}{4}$	**220594.2.a.e**
-88	**191642.2.a.c**	-61	**267851.2.a.a**	-32	**550.2.a.j**	-9	**537.2.a.a**
-84	**335118.2.a.s**	-60	**127770.2.a.c**	-31	**40331.2.a.b**	$-\frac{44}{5}$	**476410.2.a.k**
-82	**125050.2.a.s**	-59	**243611.2.a.b**	-30	**36870.2.a.e**	$-\frac{35}{4}$	**192430.2.a.h**
-81	**22353.2.a.a**	-58	**232058.2.a.c**	-29	**33611.2.a.a**	$-\frac{17}{2}$	**22406.2.a.g**
-80	**72790.2.a.g**	-57	**44175.2.a.b**	-28	**15274.2.a.c**	$-\frac{169}{20}$	**447070.2.a.k**
-76	**251218.2.a.l**	-56	**4774.2.a.i**	-27	**3075.2.a.b**	$-\frac{33}{4}$	**166650.2.a.cd**
-75	**96735.2.a.a**	-55	**199595.2.a.b**	-26	**806.2.a.d**	$-\frac{81}{10}$	**461130.2.a.w**
-74	**465386.2.a.d**	-53	**179723.2.a.a**	-25	**4495.2.a.a**	-8	**302.2.a.c**
-73	**447563.2.a.a**	-52	**85150.2.a.p**	-24	**5034.2.a.g**	$-\frac{31}{4}$	**143158.2.a.b**
-72	**35850.2.a.bb**	-51	**161211.2.a.a**	-23	**17963.2.a.a**	$-\frac{38}{5}$	**60610.2.a.be**
-71	**413291.2.a.b**	-50	**30490.2.a.a**	-22	**15950.2.a.l**	$-\frac{15}{2}$	**16530.2.a.bb**
-70	**396830.2.a.e**	-49	**20573.2.a.a**	-21	**14091.2.a.b**	$-\frac{29}{4}$	**121858.2.a.c**
-69	**380811.2.a.a**	-48	**16986.2.a.e**	-20	**6190.2.a.f**	$-\frac{36}{5}$	**97530.2.a.h**
-68	**182614.2.a.h**	-47	**128075.2.a.c**	-19	**10811.2.a.a**	-7	**175.2.a.a**

Finally, consider the 2-parameter mirror to $\begin{matrix}\mathbb{P}^2\\\mathbb{P}^2\end{matrix}\begin{bmatrix}3\\3\end{bmatrix}$.

Once again, the axiodilaton is a ratio of integrals of a product of Meijer G functions and the j-invariant of this is

$$j(\tau(\varphi)) \;=\; -\frac{(24\varphi+1)^3}{\varphi^3(27\varphi+1)} \;.$$

Tables for example 3

φ	Modular form label	φ	Modular form label	φ	Modular form label	φ	Modular form label
-100	26990.2.a.k	-79	84214.2.a.b	-51	4386.2.a.g	-23	7130.2.a.b
-99	11022.2.a.k	-78	164190.2.a.q	-50	13490.2.a.c	-22	13046.2.a.e
-98	1610.2.a.c	-77	160006.2.a.a	-49	9254.2.a.f	-21	11886.2.a.a
-97	253946.2.a.h	-76	77938.2.a.n	-48	7770.2.a.y	-20	770.2.a.g
-96	15546.2.a.b	-75	7590.2.a.m	-47	29798.2.a.a	-19	38.2.a.a
-95	121790.2.a.b	-74	147778.2.a.d	-46	57086.2.a.a	-18	2910.2.a.j
-94	238478.2.a.f	-73	143810.2.a.b	-45	18210.2.a.d	-17	7786.2.a.a
-93	233430.2.a.j	-72	11658.2.a.t	-44	26114.2.a.c	-16	862.2.a.f
-92	114218.2.a.f	-71	68018.2.a.a	-43	12470.2.a.f	-15	3030.2.a.l
-91	55874.2.a.c	-70	132230.2.a.m	-42	47586.2.a.r	-14	5278.2.a.d
-90	72870.2.a.bc	-69	18354.2.a.j	-41	45346.2.a.b	-13	910.2.a.d
-89	213778.2.a.a	-68	62390.2.a.c	-40	10790.2.a.i	-12	1938.2.a.j
-88	2090.2.a.o	-67	15142.2.a.a	-39	20514.2.a.c	-11	814.2.a.a
-87	102138.2.a.f	-66	117546.2.a.o	-38	7790.2.a.c	-10	2690.2.a.d
-86	199606.2.a.a	-65	114010.2.a.a	-37	36926.2.a.b	$-\frac{49}{5}$	230650.2.a.d
-85	194990.2.a.j	-64	3454.2.a.h	-36	5826.2.a.e	$-\frac{39}{4}$	163644.2.a.c
-84	95214.2.a.j	-63	3570.2.a.j	-35	4130.2.a.a	$-\frac{243}{25}$	122550.2.a.q
-83	5810.2.a.a	-62	103726.2.a.a	-34	31178.2.a.j	$-\frac{48}{5}$	193650.2.a.p
-82	181466.2.a.d	-61	100406.2.a.b	-33	29370.2.a.m	$-\frac{19}{2}$	38836.2.a.b
-81	6558.2.a.f	-60	48570.2.a.g	-32	1726.2.a.b	$-\frac{47}{5}$	185650.2.a.g
-80	21590.2.a.c	-59	23482.2.a.a	-31	12958.2.a.d	$-\frac{37}{4}$	147260.2.a.d

Geometric comments

We expect that an elliptic surface should be found in all examples, as was done for Example 1.

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We do not presently have a systematic realisation of the modular curve for all examples.

We have focussed here on examples where computations are the simplest, mirrors of favourable CICYs. There could be many more examples waiting in the set of mirrors to non-favourable CICYs, or indeed in members of mirror-pairs not including a CICY.

In the F-theory uplift of these flux vacua, the axiodilaton that we have computed is promoted to the modulus of an elliptic fibration, which is constant over its base. **[Kachru, Nally, Yang, 2020]**

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The j-invariants that we have computed match those of the F-theory fibres, and when a suitable choice of coordinates is made the elliptic curve related to our modular forms by the modularity theorem is precisely the F-theory curve.

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This was already the case in KNY's example, which then appears to hold in other cases as well.

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There is a surprise here, as the modularity conjecture then suggests that in supersymmetric configurations the F-theory fourfold should contain the modular curve once in the fibre and again as part of a ruled surface in the base.

Our flux vectors specify an integral lattice $\Lambda_2 \subset H^3_{\text{dR}}(\mathcal{X}, \mathbb{Z})$ such that $\mathbb{C} \otimes \Lambda_2 \subset H^{1,2}(\mathcal{X}, \mathbb{C}) \oplus H^{2,1}(\mathcal{X}, \mathbb{C})$. This provides us with a *critical elliptic motive*.

Deligne's conjecture predicts that for critical motives M, the L-value $L(M, 0)$ is a rational, possibly zero, multiple of the Deligne period $c^+(M)$,

$$\frac{L(M,0)}{c^+(M)} \in \mathbb{Q} \, .$$

L is computed from a Mellin transform of the modular form read off from our zeta numerators, and $c^+(M)$ is computed from our Calabi-Yau periods.

Thank you for listening.